THE NOETHER THEOREM FOR GEOMETRIC ACTIONS AND AREA PRESERVING DIFFEOMORPHISMS ON THE TORUS

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We find that within the formalism of coadjoint orbits of the infinite dimensional Lie group the Noether procedure leads, for a special class of transformations, to the constant of motion given by the fundamental group one-cocycle S. Use is made of the simplified formula giving the symplectic action in terms of S and the Maurer–Cartan one-form. The area preserving diffeomorphisms on the torus $T^2 = S^1 \otimes S^1$ constitute an algebra with central extension, given by the Floratos–Iliopoulos cocycle. We apply our general treatment based on the symplectic analysis of coadjoint orbits of Lie groups to write the symplectic action for this model and study its invariance. We find an interesting abelian symmetry structure of this non-linear problem.

1. The Noether theorem for symplectic actions

Recently we have formulated a new simplified method to produce general geometric actions [1–4] in closed form using the symplectic structure naturally defined on coadjoint orbits [5,6]. This has established a relation between the physical actions and the underlying geometry and group structure. The key observation made by us is that the action is given as a simple product of the one-cocycle $S(g)$ of the involved Lie group G with values in the $\mathscr{G}^*$ (a dual space to the Lie algebra $\mathscr{G}$) and the corresponding Maurer–Cartan form $\mathscr{Y}$ [7]. Hence the form of the action and its fundamental properties are totally determined in terms of the basic group theoretical objects defined entirely by the underlying Lie group G having a non-trivial central extension. This technique will enable us here to make general observations concerning invariance and constants of motion associated to the actions obtained in the above way.

Let G be a Lie group which has a non-trivial central extension $\tilde{G}$. The elements of the corresponding Lie algebra $\tilde{\mathscr{G}}$ are represented by pairs (ξ, n), where ξ is a hamiltonian function on the symplectic manifold M, which under a natural homomorphism is being mapped into the Lie algebra $\mathscr{G}$ of G, while n is a central element. The dual vector in $\tilde{\mathscr{G}}^*$ is written as (B, c). For the above dual pairs we define the following bilinear form $\langle \cdot | \cdot \rangle$:

$$\langle (\cdot, c) | (\cdot, n) \rangle = \langle \cdot | \cdot \rangle_0 + cn \,, \tag{1}$$

where $\langle \cdot | \cdot \rangle_0$ is a natural bilinear form on $\mathscr{G}$.

Let S be a one-cocycle on G taking values in $\mathscr{G}^*$ and satisfying the cocycle condition

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$$(\delta S)(g_1, g_2)$$
$$= g_1 \circ^* S(g_2) - S(g_1 g_2) + S(g_1) = 0\,, \qquad (2)$$

as well as relations $S(I) = 0$ and $S(g) = -g \circ^* S(g^{-1})$ [7].

We now set the adjoint action of $g \in \mathrm{G}$ on the (ξ, n) pair to be

$$\mathrm{Ad}_g(\xi, n) = (g \circ \xi, n + \lambda \langle S(g^{-1}) | \xi \rangle_0)\,, \qquad (3)$$

where $g \circ a$ defines the standard adjoint transformation on G.

By invariance of the bilinear form we obtain from eq. (3) the corresponding coadjoint action of $\tilde{\mathrm{G}}$

$$\mathrm{Ad}^*_g(B, c) = (g \circ^* B + c\lambda S(g), c)\,, \qquad (4)$$

where $\circ^*$ denotes the coadjoint action of G and λ is a constant. This form of the coadjoint action for $\mathrm{G} = \mathrm{SDiff\,S}^1$ has been derived in ref. [6].

The adjoint representation of the Lie group induces the adjoint representation of its Lie algebra

$$\mathrm{ad}_{(\xi_1, n_1)}(\xi_2, n_2) = [(\xi_1, n_1), (\xi_2, n_2)] \qquad (5)$$

with the commutator of the Lie algebra given by

$$[(\xi_1, n_1), (\xi_2, n_2)] = (\{\xi_1, \xi_2\}, \omega(\xi_1, \xi_2))\,, \qquad (6)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket for smooth hamiltonian functions on M. Here we use the fact that there is a natural homomorphism from the Lie algebra of the smooth functions on M equipped with the Poisson bracket into the Lie algebra $\mathscr{G}$, which is transforming $\{\cdot, \cdot\}$ into the usual Lie commutator for the vector fields in $\mathscr{G}$. This remark justifies the above use of the hamiltonian functions.

The Lie algebra cocycle $\omega(\cdot, \cdot)$ can be expressed in terms of $s(\xi)$, the infinitesimal limit of $S(g)$, as

$$\omega(\xi_1, \xi_2) = -\lambda \langle s(\xi_1) | \xi_2 \rangle_0\,. \qquad (7)$$

Using invariance of the bilinear form $\langle \cdot | \cdot \rangle$ one can find the corresponding coadjoint action

$$\mathrm{ad}^*_{(\xi,n)}(B, c) = (\mathrm{ad}^*_\xi(B) + c\lambda s(\xi), 0)\,. \qquad (8)$$

As shown in ref. [7] the one-cocycle $S(g)$ is a covector varying along the co-orbit according to

$$\mathrm{d}(S, 1/\lambda) = \mathrm{ad}^*_{(y, m_y)}(S, 1/\lambda)\,, \qquad (9)$$

where d is the exterior derivative along the co-orbit and $\mathscr{Y} = (y, m_y)$ is a Maurer–Cartan one-form on $\tilde{\mathscr{G}}$ entering the Maurer–Cartan equation

$$\mathrm{d}\mathscr{Y} = \tfrac{1}{2}[\mathscr{Y}, \mathscr{Y}]\,. \qquad (10)$$

Recalling definition (6) we find for the central element of $\mathscr{Y}$

$$m_y = \tfrac{1}{2}\mathrm{d}^{-1}\omega(y, y)\,. \qquad (11)$$

According to our result from ref. [7] the corresponding action density in the symplectic framework appears as

$$\alpha_c = -\lambda c \langle (S, 1/\lambda) | \mathscr{Y} \rangle$$
$$= -\lambda c [\langle S | y \rangle_0 + (1/\lambda) m_y]\,. \qquad (12)$$

This action defines the symplectic two-form Ω_c by acting with the exterior derivative d on α_c:

$$\Omega_c \equiv \mathrm{d}\alpha_c = -\tfrac{1}{2}\lambda c \langle \mathrm{d}(S, 1/\lambda) | \mathscr{Y} \rangle$$
$$= -\tfrac{1}{2}\lambda c \langle \mathrm{d}S | y \rangle_0\,. \qquad (13)$$

We will now show that for g transforming under the left multiplication $g \to (1+\epsilon) g$, with $S(g) = S^i$ transforming according to the coadjoint action

$$\delta_\epsilon(S, 1/\lambda) = (S^f, 1/\lambda) - (S^i, 1/\lambda)$$
$$= \mathrm{ad}^*_{(\epsilon,n)}(S, 1/\lambda) = (\mathrm{ad}^*_\epsilon(S) + s(\epsilon), 0)\,, \qquad (14)$$

as follows from (2), the action α_c transforms as

$$\alpha_c \to \alpha_c - \tfrac{1}{2}\lambda c \langle S | \mathrm{d}\epsilon \rangle_0\,. \qquad (15)$$

Accordingly, S is a constant of motion for this type of transformations, as follows from this adapted version of the Noether theorem.

To prove it we first find the transformation $\delta_\epsilon y = y^f - y^i$ of the Maurer–Cartan form corresponding to (14). Consider first

$$\langle \mathrm{ad}^*_{(y^f, m_y)}(S^f, 1/\lambda) | X \rangle = -\langle (S^f, 1/\lambda) | [\mathscr{Y}^f, X] \rangle$$
$$= -\langle (S^i, 1/\lambda) | [\mathscr{Y}^i, X] \rangle$$
$$- \langle (S^i, 1/\lambda) | [\delta_\epsilon \mathscr{Y}, X] \rangle$$
$$- \langle \delta_\epsilon(S, 1/\lambda) | [\mathscr{Y}^i, X] \rangle + \mathrm{O}(\epsilon^2)\,; \qquad (16)$$

therefore

$$\langle \mathrm{ad}^*_{y^f}(S^f, 1/\lambda) | X \rangle - \langle \mathrm{ad}^*_{y^i}(S^i, 1/\lambda) | X \rangle$$
$$= -\langle (S^i, 1/\lambda) | [\delta_\epsilon \mathscr{Y}, X] \rangle$$
$$+ \langle (S^i, 1/\lambda) | [(\epsilon, n), [\mathscr{Y}^i, X]] \rangle\,. \qquad (17)$$

Using relation (9) we can alternatively rewrite the left-hand side of the above equation as

$$\begin{aligned}&\langle \mathrm{d}(S^{\mathrm{f}}, 1/\lambda) - \mathrm{d}(S^{\mathrm{i}}, 1/\lambda) \,|\, X\rangle \\ &= -\langle \mathrm{ad}^*_{\mathrm{d}(\epsilon,n)}(S^{\mathrm{i}}, 1/\lambda) + \mathrm{ad}^*_{(\epsilon,n)}\mathrm{d}(S^{\mathrm{i}}, 1/\lambda) \,|\, X\rangle \\ &= \langle (S^{\mathrm{i}}, 1/\lambda) \,|\, [\mathrm{d}(\epsilon, n), X]\rangle \\ &\quad - \langle (S^{\mathrm{i}}, 1/\lambda) \,|\, [\mathscr{Y}^{\mathrm{i}}, [(\epsilon, n), X]]\rangle . \end{aligned} \tag{18}$$

Comparing eqs. (17) and (18) and making use of the Jacobi identity yields a transformation formula

$$\delta_\epsilon y = \{\epsilon, y\} + \mathrm{d}\epsilon , \tag{19}$$

which should be understood in a weak sense. Accordingly Ω_{c} defined in (13) transforms (for ϵ depending on the co-orbit parameter t) as

$$\delta_\epsilon \Omega_{\mathrm{c}} = -\tfrac{1}{2}\lambda c(\langle \delta_\epsilon \mathrm{d}S | y\rangle_0 + \langle \mathrm{d}S | \delta_\epsilon y\rangle_0) . \tag{20}$$

Since the above derivative d must act in the direction perpendicular to t we obtain by commuting d with δ_ϵ

$$\delta_\epsilon \Omega_{\mathrm{c}} = -\tfrac{1}{2}\lambda c\langle \mathrm{d}S | \mathrm{d}\epsilon\rangle_0 , \tag{21}$$

which in consequence leads to the desired form for the variation of the action density

$$\delta_\epsilon \alpha_{\mathrm{c}} = -\tfrac{1}{2}\lambda c\langle S | \mathrm{d}\epsilon\rangle_0 . \tag{22}$$

This concludes our proof.

A particularly transparent example of this method is provided by the infinite dimensional Lie group $G = \mathrm{Diff}\, S^1$. In this case the schwartzian derivative $S(F) = F'''/F' - \frac{3}{2}(F''/F')^2$ is the one-cocycle, while the Maurer–Cartan one-form is given by $y = \mathrm{d}F/F'$. We study in this case the reparametrization $x \to x + \epsilon$ for $x \in S^1$. The corresponding variation of $F \in \mathrm{Diff}\, S^1$ is given as $\delta_\epsilon F = \epsilon(\partial/\partial x)F(x)$. This variation corresponds to the action of the vector field $\epsilon\partial_x$ on $F(x)$ from the left and therefore S transforms according to (14) [7]. Our proof dictates then the transformation rule of y as in (19), which in this case can easily be verified explicitly. The schwartzian derivative is therefore a constant of motion for the action density α_{c} [2,8].

2. The area preserving diffeomorphisms on the torus

Here we apply the formalism of the first section to the area preserving diffeomorphisms on the torus $T^2 = S^1 \otimes S^1$ and present the corresponding field theory.

Let us first recall a well-known result from symplectic mechanics, namely that the condition for a field $\boldsymbol{v}$ to be a local hamiltonian field in two dimensions is $\mathrm{div}\, \boldsymbol{v} = 0$. This result specially holds for $\boldsymbol{v}$ having a physical interpretation as a velocity field for the flow of liquid of constant density [9,10]. Consequently the field $\boldsymbol{v}$ is determined by some hamiltonian function g such that $\boldsymbol{v} = \mathrm{rot}\, g$ and the vector field can be cast in the form

$$L_g \equiv \frac{\partial g}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial g}{\partial x_2}\frac{\partial}{\partial x_1} = v_i \partial_i \tag{23}$$

with $g = g(x_1, x_2)$ defined on the torus $(0, 2\pi) \times (0, 2\pi)$. For the hamiltonian functions on the torus we introduce the Poisson structure through the natural Lie algebra homomorphism $g \to L_g$ satisfying

$$[L_g, L_f] = L_{\{g,f\}} \tag{24}$$

with the Poisson bracket

$$\{g, f\} = \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2} - \frac{\partial g}{\partial x_2}\frac{\partial f}{\partial x_1} = \epsilon^{ij}\partial_i g \partial_j f . \tag{25}$$

Hence the Poisson bracket $\{g, f\}$ is chosen in such a way as to ensure it is being mapped by $g \to L_g$ into the usual vector field Lie commutator of L_g and L_f.

For the periodic basis functions $g(x_1, x_2) = \exp[\mathrm{i}(n_1 x_1 + n_2 x_2)]$ the vector field commutator becomes [11]

$$[L_{\boldsymbol{n}}, L_{\boldsymbol{m}}] = -\boldsymbol{n} \times \boldsymbol{m}\, L_{\boldsymbol{n}+\boldsymbol{m}} + a_i n_i \delta_{\boldsymbol{n}+\boldsymbol{m},0} , \tag{26}$$

where we included the non-trivial central term, which as shown in ref. [11] is admitted by this algebra, with a_1 and a_2 being some constants. The area preserving algebra of the torus and its trigonometric analogue have recently received a lot of attention in the literature [12]; here we will discuss the corresponding group structure in the symplectic setting.

The algebra structure (26) is induced by two-dimensional reparametrizations $x_1 \to \sigma_1(x_1, x_2)$ and $x_2 \to \sigma_2(x_1, x_2)$ with the area preserving condition

$$\det \frac{(\sigma_1, \sigma_2)}{(x_1, x_2)} = \{\sigma_1, \sigma_2\} = 1 , \tag{27}$$

which ensures that the Poisson structure (25) is preserved.

One can verify that the above determinant condition imposes the following relations:

$$\frac{\partial\sigma_k}{\partial x_k}=\frac{\partial x_i}{\partial\sigma_i}\,,\quad \frac{\partial\sigma_i}{\partial x_k}=-\frac{\partial x_i}{\partial\sigma_k} \tag{28}$$

for fixed $i\neq k$.

The area preserving reparametrizations define the infinite dimensional Lie group $G=\mathrm{SDiff}\,T^2$ and induce the following adjoint and coadjoint transformations:

$$g\circ\xi=\xi(\sigma_1(x_1,x_2),\sigma_2(x_1,x_2))\,,$$
$$g\circ^* b=b(\sigma_1(x_1,x_2),\sigma_2(x_1,x_2)) \tag{29}$$

by $g\in\mathrm{SDiff}\,T^2$.

To ξ in the algebra $\mathscr{G}\equiv\mathrm{Vect}\,T^2$ one associates as in the first section the pair (ξ,n) in $\tilde{\mathscr{G}}$ – the central extension of Vect T^2. The commutator structure on $\tilde{\mathscr{G}}$ is given by eq. (6) with the Poisson bracket as in (25) and with ω being the Floratos–Iliopoulos cocycle

$$\omega(\xi,\eta)=-\int\!\!\int \mathrm{d}x_1\,\mathrm{d}x_2 a_i\partial_i\xi\,\eta\,. \tag{30}$$

With the choice of the constant λ to be one, (30) implies that s (the infinitesimal limit of the one-cocycle S) is $s(\xi)=a_i\partial_i\xi$. In this setting the infinitesimal coadjoint transformation (8) takes the form

$$\mathrm{ad}^*_{(\xi,n)}(B,c)=(\{\xi,B\}+ca_i\partial_i\xi,0)\,. \tag{31}$$

From (27) it follows that

$$\{\mathrm{d}\sigma_1,\sigma_2\}+\{\sigma_1,\mathrm{d}\sigma_2\}=0\,, \tag{32}$$

which can be rewritten as $\mathrm{div}(\mathrm{d}\boldsymbol{\sigma})=0$. Locally we therefore can find the one-form y satisfying $\mathrm{d}\boldsymbol{\sigma}=\mathrm{rot}\,y$ or explicitly

$$\frac{\partial y}{\partial\sigma_2}=-\mathrm{d}\sigma_1\,,\quad \frac{\partial y}{\partial\sigma_1}=\mathrm{d}\sigma_2\,. \tag{33}$$

Accordingly, the one-form y is a hamiltonian for the vector field $\mathrm{d}\sigma_i$ and plays an important role in our formalism. Let us namely consider an arbitrary covector $U_0(x_1,x_2)$. Applying the adjoint transformation we get $U(\sigma_1,\sigma_2)=g\circ^* U_0$. We have

$$\mathrm{d}U=\frac{\partial U}{\partial\sigma_1}\mathrm{d}\sigma_1+\frac{\partial U}{\partial\sigma_2}\mathrm{d}\sigma_2=\{y,U\}\,, \tag{34}$$

which follows from the definition of the Poisson bracket (25) and property (33). Hence y is a solution to the basic equation [2,7]

$$\mathrm{d}(\mathrm{Ad}^*_g U_0)=\mathrm{ad}^*_y(\mathrm{Ad}^*_g U_0)\,, \tag{35}$$

which leads us to consider y as a Maurer–Cartan form satisfying

$$\mathrm{d}y=\tfrac{1}{2}\{y,y\}=\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,, \tag{36}$$

where we used again (25) and (33). A general solution to the above equation is of the form

$$y=\tfrac{1}{2}(\sigma_1\mathrm{d}\sigma_2-\sigma_2\mathrm{d}\sigma_1)+\mathrm{d}\rho\,, \tag{37}$$

where the one-form $\mathrm{d}\rho$ expresses the fact that the Maurer–Cartan equation (36) specifies y only modulo an exact form. It remains to be checked whether the above solution (37) satisfies the basic equation (33). It is convenient at this point to rewrite (33) as

$$\partial_m y=\epsilon_{ij}\partial_m\sigma^i\mathrm{d}\sigma^j\,, \tag{38}$$

where ∂_m denotes a derivative with respect to x^m. Applying ∂_m on both sides of (37) and using the area preserving condition (27) we find that consistency requires

$$\partial_m\rho=-\tfrac{1}{2}\epsilon_{ij}\sigma^i\partial_m\sigma^j-\tfrac{1}{2}\epsilon_{mn}x^n\,. \tag{39}$$

With this condition imposed on ρ the solution (37) to the Maurer–Cartan equation becomes fully compatible with the interpretation of y as a hamiltonian for $\mathrm{d}\sigma^i$ given in (33).

We can now use the Maurer–Cartan form y to determine the one-cocycle S according to (9) which is explicitly, in this case, given by

$$\begin{aligned}\mathrm{d}(S,1)&=\mathrm{ad}^*_{(y,m_y)}(S,1)\\&=(\{y,S\}+a_i\partial_i y,0)\,.\end{aligned} \tag{40}$$

We take the ansatz $S=T(\sigma_1,\sigma_2)+F(x_1,x_2)$. It follows that because of eq. (34) we only have to solve for the x-dependent part of (40), which is

$$\{y,F(x)\}+a_i\partial_i y=0\,. \tag{41}$$

Using the identity $s(\xi)=-a_i\epsilon^{ij}\{\xi,x_j\}$ we find that $F(x)=a_2x_1-a_1x_2$. The boundary condition $S(x_i=\sigma_i)=0$ imposes $T(\sigma)=a_1\sigma_2-a_2\sigma_1$. In conclusion the one-cocycle of $\mathrm{SDiff}\,T^2$ is given by

$$\begin{aligned}S(x\to\sigma)&=T(\sigma)+F(x)\\&=a_1(\sigma_2-x_2)-a_2(\sigma_1-x_1)\,,\end{aligned} \tag{42}$$

and one easily verifies that $S(x\to\sigma)$ indeed satisfies the cocycle condition (2). An alternative and interesting derivation of the group one-cocycle directly from the Lie algebra cocycle ω has been provided by Kirillov [6]. Starting with the general assumptions $H^1(\mathscr{G})=0$ and $\dim H^2(\mathscr{G})=1$ about the homologies of the infinite dimensional Lie algebra $\mathscr{G}$, Kirillov has found that the one-cocycle can be obtained through the formula

$$\omega(\xi,\eta)-L(g)\omega(\xi,\eta)=\langle S|\,[\xi,\eta]\,\rangle_0\,, \tag{43}$$

where $L(g)\omega(\xi,\eta)=\omega(g\circ\xi, g\circ\eta)$.

We will illustrate Kirillov's method for $\mathscr{G}=\mathrm{Vect}\,T^2$. Observe first that

$$L(g)\omega(\xi,\eta)=\int\!\!\int d\sigma_1\,d\sigma_2\,a_i\frac{\partial\xi}{\partial\sigma_k}\frac{\partial\sigma_k}{\partial x_i}\eta\,; \tag{44}$$

substituting (28) and integrating by parts we find

$$\begin{aligned}&\omega(\xi,\eta)-L(g)\omega(\xi,\eta)\\&=\int\!\!\int d\sigma_1\,d\sigma_2\left((-a_1x_2+a_2x_1)\{\xi,\eta\}\right.\\&\left.-a_1\frac{\partial\xi}{\partial\sigma_1}\eta-a_2\frac{\partial\xi}{\partial\sigma_2}\eta\right)\\&=\int\!\!\int d\sigma_1\,d\sigma_2\{\xi,\eta\}S(x\to\sigma)\,,\end{aligned} \tag{45}$$

reproducing (42).

Having determined both S and y we can now proceed with the calculation of the action density given in formula (12). For this purpose we now find the central element m_y of the extended Maurer–Cartan form $\mathscr{Y}=(y, m_y)$. Inserting the Floratos–Iliopoulos cocycle (30) into the definition (11) we obtain

$$dm_y=-\tfrac{1}{2}\int\!\!\int dx_1\,dx_2\left(a_1\frac{\partial y}{\partial x_1}y+a_2\frac{\partial y}{\partial x_2}y\right). \tag{46}$$

Substituting $\partial y/\partial x_i=(\partial y/\partial\sigma_k)\partial\sigma_k/\partial x_i$ and using eqs. (33) and (28) we arrive at

$$\begin{aligned}dm_y=-\tfrac{1}{2}\int\!\!\int dx_1\,dx_2&\left[a_1\left(d\sigma_2\frac{\partial x_2}{\partial\sigma_2}+d\sigma_1\frac{\partial x_2}{\partial\sigma_1}\right)\right.\\&\left.-a_2\left(d\sigma_2\frac{\partial x_1}{\partial\sigma_2}+d\sigma_1\frac{\partial x_1}{\partial\sigma_1}\right)\right]y\,.\end{aligned} \tag{47}$$

Performing now an integration by parts and recalling that $\mathrm{div}(d\boldsymbol{\sigma})=0$ as well as (33) we obtain

$$\begin{aligned}m_y&=\tfrac{1}{2}\int\!\!\int dx_1\,dx_2(a_1x_2-a_2x_1)(\sigma_1d\sigma_2-\sigma_2d\sigma_1)\\&=\tfrac{1}{2}\int\!\!\int dx_1\,dx_2\,F(x)(\sigma_1d\sigma_2-\sigma_2d\sigma_1)\,.\end{aligned} \tag{48}$$

We are now ready to calculate the action density (12) $\alpha=-(\langle S|y\rangle_0+m_y)$, where we have omitted the central element c of the covector since it can always be absorbed in the constants a_i. As the corresponding symplectic two-form Ω is only defined up to the exact form we obtain the following nontrivial contribution to the action density:

$$\alpha=-\tfrac{1}{2}\int\!\!\int dx_1\,dx_2T(\sigma)(\sigma_1d\sigma_2-\sigma_2d\sigma_1)\,, \tag{49}$$

where we have used the fact that $\langle S|d\rho\rangle_0=-\tfrac{5}{3}\langle T|\tfrac{1}{2}(\sigma_1d\sigma_2-\sigma_2d\sigma_1)\rangle$ up to the closed form, for $d\rho$ defined in (37).

The complete action can be written (modulo exact terms and constants in front) as

$$\mathscr{A}=\int\!\!\int\!\!\int dx_1\,dx_2(a_1\sigma_2^2d\sigma_1+a_2\sigma_1^2d\sigma_2)\,, \tag{50}$$

where the first integral is over a curve on the orbit parametrized by t.

Now let us study the effect of space reparametrizations

$$\begin{aligned}&x_1\to x_1+\epsilon_1(x_1,x_2,t)\,,\\&x_2\to x_2+\epsilon_2(x_1,x_2,t)\end{aligned} \tag{51}$$

on the action (50). The infinitesimal version of the determinant condition (27) will be satisfied by setting $\epsilon_i=-\epsilon^{ij}\partial_j\epsilon$, with an arbitrary infinitesimal function $\epsilon(x_1, x_2, t)$. The action (50) varies as follows under (51):

$$\delta_\epsilon\mathscr{A}=-2\int\!\!\int\!\!\int dx_1\,dx_2T(\sigma)\,d\epsilon\,, \tag{52}$$

where we have used integration by parts and the determinant condition (27). In conclusion we find that the corresponding constant of motion is $T(\sigma)=a_1\sigma_2-a_2\sigma_1$. This is in total agreement with the general observations made in the previous section. $T(\sigma)$ transforms as follows under the reparametrizations (51):

$$\delta_\epsilon T=\{\epsilon, T\}=a_i\frac{\partial\epsilon}{\partial\sigma_i}\,. \tag{53}$$

Applying transformation rules (52) and (53) to the appropriate correlation function we obtain a central extension of the abelian algebra for the commutators

$$[T_{\boldsymbol{m}}, T_{\boldsymbol{n}}] = \boldsymbol{a} \cdot \boldsymbol{n}\, \delta_{\boldsymbol{m}+\boldsymbol{n},0}\,, \tag{54}$$

where $T(\sigma) = \sum_{\boldsymbol{m}} T_{\boldsymbol{m}} \exp(\mathrm{i}\boldsymbol{m} \cdot \boldsymbol{\sigma})$.

The invariance of the geometric actions (12) is related to the isotropy group, which is a subgroup of G, which leaves the one-cocycle $S(g)$ invariant. In the case of $\mathrm{G} = \mathrm{SDiff}\, \mathrm{T}^2$ it is enough to consider the transformations which leave $T(\sigma)$ invariant. The form of these must therefore be $\sigma_i \to \sigma_i + f_i$ with the functions f_i satisfying $a_2 f_1 = a_1 f_2$. The determinant condition (27) imposes on f_1 the relation $a_1 \partial f_1 / \partial \sigma_1 - a_2 \partial f_1 / \partial \sigma_2 = 0$, which classically can be rewritten as a Poisson bracket $\{T, f_1\} = 0$. The general solution to this constraint must therefore be $f_1 = f(T)$. Hence it is enough to consider transformations of the type $f_1 = \lambda T^n$, $f_2 = \lambda (a_2/a_1) T^n$, with λ being an infinitesimal time dependent parameter. A tedious but straightforward calculation yields the following result for the variation of the action:

$$\delta_\lambda \mathscr{A} = 2 \int \int \int \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\mathrm{d}\lambda}{(n+2)a_1} T^{n+2}\,, \tag{55}$$

giving the conservation laws $(\mathrm{d}/\mathrm{d}t) T^{n+2} = 0$ ($n = 0, 1, 2, \ldots$). We find therefore an infinite number of constants of motion entering an abelian algebra with a central extension. For a classical abelian algebra (without the central charge), the constants of motion would form a complete set of integrals of motion in involution and the dynamical system under consideration would be completely integrable – a situation familiar from $(1+1)$ dimensions.

In conclusion, find that the area preserving diffeomorphism on the torus possesses a feature characteristic for the two-dimensional models, namely that its central element uniquely determines the content of the model itself. One observes that this is associated with the fundamental property of the torus: not being simply connected. For the area preserving diffeomorphisms on the plane $\mathrm{SDiff}\,\mathbb{R}^2$ [13] one finds namely easily that the algebra does not allow any central extension.

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